Indian J. Phys. 43, 468—473 (1969)

Variation of modulus of elasticity with frequency of vibration

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(*Received 22 August* 1969)

Young's moduli of various species of wood at different frequencies have been determined experimentally by Kakkar (1969). It has been found that there is a gradual increase of Young's Modulus for a particular species with increase in frequency leading to a constant value at high frequency. The variation has been explained on the assumption that the state of vibration of the specimen gradually changes from isothermal to adiabatic condition. An expression has been derived which agrees fairly well with experimentally observed variation in the case of wood and perspex.

Introduction

Experimental beam was placed horizontally on two knife edges separated by a known distance in such a way that their positions formed the nodes of the vibrating beam (Free-Free bar method). By changing the length of the specimen it was made to vibrate at different resonant frequencies with the help of an audio oscillator feeding a dc energised electromagnet (Kakkar 1969). From the knowledge of the resonant frequency the dynamic Young's modulus was calculated, taking a small depth span ratio. (Lord Rayleigh 1926, Timoshenko 1921, 1922).

It was found that the value of dynamic Young's modulus for wooden species continually rises with frequency of vibration to a limiting value, which is 20 to 25% higher than the static value.

Theoretical

The higher value of dynamic modulus of elasticity under vibrating condition has been accounted for as due to the thermal effect in the filament.

When a solid is compressed, mechanical energy is partly converted into thermal energy and the compressed parts become warmer than the extended parts. Energy transfer between elements with a temperature difference results in a different elastic constant for adiabatic conditions (no heat lost from the compressed region) than for isothermal conditions (temperature equilibrium obtains between all the stressed parts).

From Maxwell's second relation of thermodynamics, if an element is suddenly stretched (extended) or compressed by applying a force ΔF, ΔT a cooling or heating effect takes place in the element given by the formula (Saha & Srivastava 1950)

$$\Delta T = \frac{T\alpha}{WC}\,\Delta F \qquad \ldots(1)$$

where α = Positive coefficient of linear expansion.
W = mass per unit length of the element
C = Specific heat
T = room temperature in absolute degrees.

Since in the case of a vibrating beam extensions and compressions take place in quick succession, equation (1) may usefully be applied to evaluate the heating or cooling produced in the case of a vibrating beam and hence to explain the frequency dependence of modulus of elasticity.

Let us consider the instantaneous case of the compressed filament of a vibrating beam. The heat produced in a filament is used up in two ways :-

...(1) a part of heat produced is conducted away to the neighbouring cooler filaments.

(2) the remaining part is confined to the filament and raises its temperature. The rise of temperature will depend on the thermal capacity of the filament.

The increased rate of vibration of a beam decreases the probability of equalisation of temperature of different filaments of the specimen because of the reduced time for conduction of heat. Progressively larger part of the heat, produced during compression, is retained by the filament itself. This state of vibration of the beam represents the gradual transition from isothermal (Static) to the adiabatic state of vibration. After a certain frequency of vibration, characteristic of the specimen and above; the loading and unloading rate is high enough to be substantially adiabatic, hence no energy is lost. This represents the state of equilibrium of the vibrating bar with the ambient medium. For a bad conductor the perfect adiabatic state of vibration would approach comparatively at low frequency than for a good conductor. If T_m represents the steady temperature of the filament (adiabatic state) and T' the temperature at any lower frequency n, then we have the relation

$$-\frac{dT'}{dn} \propto \frac{T_m - T'}{KS} = \frac{\lambda}{KS}\left(T_m - T'\right) = \mu\left(T_m - T'\right) \qquad \ldots(2)$$

embodying that the rate of rise of temperature of a filament with frequency is directly proportional to the difference of maximum temperature (T_m corresponding to steady state) and the temperature of the filament T' correspomding to the frequency of vibration n and inversely proportion to the thermal conductivity (K) and thermal capacity (S) of the filament.

Here λ the constant of proportionality is independent of the nature of the material of the vibrating beam, and μ is a constant depending upon the nature of the material of the vibrating beam.

Integrating (2) and evaluating the constant of integration by applying the limiting condition $n \to 0$, $T' \to T_s$; where T_s is the instantaneous temperature of the filament in the static compressed state, we get

$$T' = T_s + (T_m - T_s)(1 - e^{-\mu n}) \quad \ldots(3)$$

Putting $T_m - T_s = T_d$ (difference of maximum and minimum temperature) we get

$$T' = T_s + T_d (1 - e^{-\mu n}) \quad \ldots(4)$$

Making use of equation (1) and considering the case of an element of unit length and unit cross section corresponding to a frequency of vibration n, the equation (4) reduces to

$$\frac{T\alpha}{\rho c} F = T_s + T + T_d (1 - e^{-\mu n}) \quad \ldots(5)$$

where ρ = density of the material.

Even at low rates of loading there are important time effects (Richards 1961). The first is the elastic after effect, or delayed elasticity, a transient variation of strain with time. One of the simpler elastic after effects is thermoelastic action. When a member is compressed, its volume decreases accompanied by a rise in its temperature. If compression takes place slowly enough so that thermal equilibrium is maintained, dissipation of heat to its surroundings holds the temperature of the member constant and the process is isothermal. If, on the other hand, compression is rapid and there is insufficient time for heat transfer, the process is adiabatic, and the temperature of the member is raised. If, after a member is loaded adiabatically to certain stress, the stress is held constant while the member is allowed to a dissipate heat, a further contraction takes place until the total contraction is the same as in the isothermal process. It will be clear from the foregoing that for the same stress, the strain will be different at different rates of vibration of the beam. Isothermal strain will be larger than the adiabatic strain. Strain will be smaller at higher rates of vibration and vice versa.

If l_0 is the maximum strain for a given stress at $n = 0$, then the strain at a frequency of vibration n will be $(l_0 - \Delta ln)$ where Δln represents the small decrease in the strain. Hence the dynamic Young's modulus at the frequency of vibration n will be given by

$$Y_n = \frac{F}{(l_0 - \Delta ln)} \quad \ldots(6)$$

From (5) and (6), we get

$$\frac{T\alpha}{\rho c} Y_n = \frac{1}{(l_0 - \Delta ln)} \left[T_s - T + T_d (1 - e^{-\beta n}) \right]$$

Since $\Delta ln \ll l_0$, neglecting higher powers of Δln we have

$$Y_n = \frac{\rho c}{T\alpha} \left(\frac{l_0 + \Delta ln}{l_0^2} \right) [T_s - T + T_d (1 - e^{-\beta n})]$$

$$= \frac{\rho c}{T\alpha l_0} T_s - \frac{\rho c}{\alpha l_0} + \frac{\rho c}{T\alpha l_0^2} \Delta ln T_s - \frac{\rho c}{\alpha l_0^2} \Delta ln$$

$$+ \frac{\rho c}{T\alpha} \left(\frac{l_0 + \Delta ln}{l_0^2} \right) T_d (1 - e^{-\beta n}) \qquad \ldots(7)$$

Since T_s is very nearly equal to T, we may take

$$\frac{\rho c}{\alpha l_0^2} \Delta ln \frac{T_s}{T} = \frac{\rho c}{\alpha l_0^2} \Delta ln$$

Then equation (7) reduces to

$$Y_n = \frac{\rho c}{\alpha l_0} \frac{T_s}{T} - \frac{\rho c}{\alpha l_0} + \frac{\rho c T_d}{T\alpha} \left(\frac{l_0 + \Delta ln}{l_0^2} \right) (1 - e^{-\beta n}) \quad (8)$$

From equation (7), when $n \to 0$, $Y_n \to Y_0$, hence we get

$$Y_0 = \frac{\rho C}{T \alpha l_0} (T_s - T) \qquad \ldots(9)$$

Here Y_0 represents the static value of Young's modulus.

Again from equation (7), when $n \to \infty$, $Y_n = Y_m$ where Y_m represents the max. value of dynamic modulus of elasticity.

$$Y_m = \frac{\rho c}{T \alpha l_0^2} (l_0 + \Delta l_m) [T_s - T + T_d] \qquad \ldots(10)$$

Here Δl_m represents the maximum decrease in strain corresponding to adiabatic state of vibration.

Let $Y_m - Y_0 = Y_d$ (the difference between the maximum value of the dynamic Young's modulus and static Young's modulus).

From (9) and (10), we get

$$Y_m - Y_0 = Y_d = \frac{\rho c}{T \alpha l_0} T_s + \frac{\rho c \Delta l_m T_s}{T \alpha l_0^2} - \frac{\rho c}{T \alpha l_0} - \frac{\Delta l_m \rho c}{\alpha l_0^2}$$

$$+ \frac{\rho c}{T \alpha l_0} T_d + \frac{\Delta l_m \rho c T_d}{T \alpha l_0^2} - \frac{\rho c}{T \alpha l_0} T_s + \frac{\rho c}{\alpha l_0} .$$

Since $\frac{T_s}{T}$ is very nealy equal to unity, the above equation reduces to

$$Y_d = \frac{\rho c}{T \overline{l_0}^2} \left[l_0 T_d + \triangle l_m T_d \right] \qquad \ldots(11)$$

From (8), (9) and (11) and taking $\triangle l_m T_d = \triangle ln\, T_d$ for various values of n, we get

$$Y_n = Y_0 + Y_d \left(1 - e^{-bn} \right) \qquad \ldots(12)$$

Discussions

By substituting the values of Y_m, Y_d and n in equation (12) from the experimentally observed data in the case of wood (Kakker 1969) the value

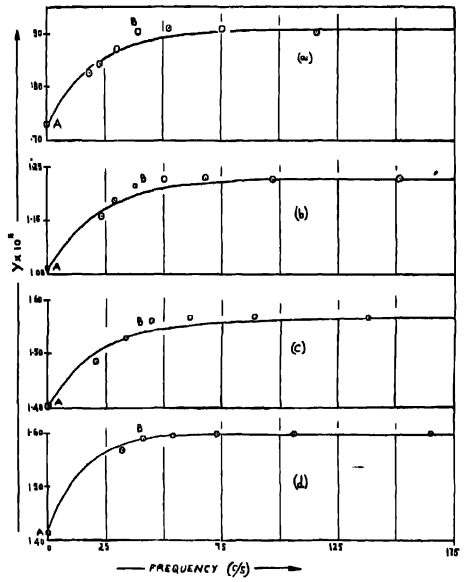

Figure 1. Variation of dynamic Young's modulus with frequency
(a) Petrocarpus Marsupium (Bija)
(b) Tectona grandis (Teak)
(c) Shores robusta (Sal).
(d) Terminalia (Saja)
A static Young's modulus
AB transition region
⊙ observed results.

of μ calculated in the transition region AB, (figure 1) for different species of wood, has been shown in column 4 of table 1.

TABLE 1

Sl. No.	Name of species	Frequency	Value of μ	Mean value of μ
1	2	3	4	5
1.	Petrocarpus Marsupium	18	0.0450	
	(Bija)	22—23	0.0488	0.049
		29	0.0518	
2.	Tectonagrandis	23	0.0385	0.0442
	(Teak)	29	0.0498	
3.	Shorearobusta	27	0.0422	0.0421
	(Sal)	34—35	0.0419	
4.	Terminalia	32—33	0.0561	0.0632
	(Saja)	41—42	0.0703	

The value of μ for a particular species at various frequencies of vibration in the transition region comes out to be nearly constant (column 4, table 1) and this is in accordance with the theoretical consideration.

Taking the average value (Kaye & Laby) of K, ρ and C :0.5 $\times$ 10^{-3} cal. cm^{-1} sec^{-1} $°C^{-1}$, 0.701 gm/cc and 0.5 cal. gm^{-1}. $°C^{-1}$ respectively and average value of μ (table 1) for the tested wooden species, λ comes out to be 8.67 $\times$ 10^{-6}. According to theoretical considerations as outlined above, λ should be the same for all materials. Later work carried out in these laboratories on perspex shows that value of λ for perspex is of the same order as for wood. This gives additional support to the correctness of underlying physical considerations in the derivation of the formula. Work with other bad and good conductors is in progress.

We are indebted to Principal V.V. Sarvate for encouragement.

REFERENCES

Kakkar, S.S. 1969 *Indian Forester*, **95**, 418.

Kaye, G. W. C. & Laby, T. H *Tables of Physical and Chemical Constants and Some Mathematical Functions* III Ed. Longman Green & Co., London, 1911-68.

Lord Rayleigh 1926 *Theory of Sound* II Ed. Vol. I, Macmillan & Co., London, 242-306

Richards Cedic, W. 1961 *Engg. Materials Science*, Wardswarth Publishing Co. Inc., San Francisco 243-246

Saha, M.N. & Srivastava, B.N. 1950 *A Treatise on Heat* III Ed. The Indian Press Ltd., Allahabad and Calcutta, 328-330.

Timoshenko, S. 1921 *Phil. Mag. Series* 6, **41**, 744.
1922 *Phil. Mag. Series* 6, **43**, 125.